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PROCESSOR STRUCTURE FOR THE DETECTION OF A SINUSOID SIGNAL OBSCURED BY WHITE NOISE

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PROCESSOR STRUCTURE FOR THE DETECTION OF A SINUSOID SIGNAL OBSCURED BY WHITE NOISE

by

W. S. Hodgkiss

Fleet Engineering Department

August 1976

NAVAL UNDERSEA CENTER
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SAN DIEGO, CALIFORNIA

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Currently employed methods for the detection of a sinusoid obscured by white noise are related to three classical problems of detection theory. The discrete Fourier transform, the power spectrum, and the averaged power spectrum each are shown to be the optimal processor (in the sense of making a least-risk decision) for a particular signal model.

SUMMARY

PROBLEM

Detection of a sinusoid obscured by white noise.

RESULTS

1. Currently employed methods for the detection of a sinusoid obscured by white noise were related to three classical problems of detection theory. The discrete Fourier transform, the power spectrum, and the averaged power spectrum each were shown to be the optimal processor (in the sense of making a least-risk decision) for a particular signal model.

2. The first- and second-order statistics under H_1 (signal plus noise) and H_0 (noise alone) were compiled for several related processing structures of current interest. Included were the autocorrelation function and the recursive exponential correlator.

RECOMMENDATIONS

1. Extend the ROC (receiver operating characteristic) performance results of this report to include the autocorrelation function and recursive exponential correlator processing structures.

2. Investigate performance via the ROC curve of the discrete Fourier transform, autocorrelation function, and recursive exponential correlator processing structures when the sinusoid's frequency varies over the observation interval.

CONTENTS

PROCESSOR STRUCTURES FOR THE DETECTION OF A SINUSOID OBSCURED BY WHITE NOISE

I. INTRODUCTION

This report has two main thrusts. Primarily, the desire is to relate currently employed methods for the detection of a sinusoid obscured by white noise to three classical problems of detection theory. It will be shown that the discrete Fourier transform, the power spectrum, and the averaged power spectrum are each the optimal processor (in the sense of making a least-risk decision) for a particular signal model. The utilization of one of these processors implicity assumes that particular signal model to exist and thus defines an upper bound on performance even if the actual received signal process could admit a processing scheme whose performance would be superior. Of secondary importance, the first- and second-order statistics under H_1 (signal plus noise) and H_0 (noise alone) are compiled for several related processing schemes of current interest. Included are the autocorrelation function and the recursive exponential correlator. These statistics will prove useful in predicting the performance of such schemes in light of the processor structures already mentioned.

Our fundamental goal is the detection of a sinusoid obscured by white noise. If the form of the received signal was known exactly, the optimal detector would simply threshold the output of a filter matched to the signal waveform. Unfortunately, the actual situation encountered is one where phase, amplitude, and frequency uncertainty usually exists.

A commonly employed processor structure for detection is a display of the discrete Fourier transform magnitude squared ($|DFT|^2$) of the observed time series (i.e., the power spectrum). In addition, averaging of the power spectra from several consecutive data sets often is performed either visually or automatically to increase the detectability of particularly weak signals.

This report will discuss several potential configurations for the detector structure. The reference point for their comparison will be the following two hypothesis testing problem:

$$\begin{aligned} H_1:&\ X(n) = S(n) + n(n) \\ H_0:&\ X(n) = n(n), \end{aligned} \tag{1.1}$$

where:

$0 \leq n \leq N-1$

$x(n)$ the observed sequence

$S(n) = A \cos(2\pi f_0 n\Delta + \phi)$ the signal sequence

$n(n)$ the white Gaussian noise sequence; $n(n) \sim N(0,\sigma^2)$.

The assumption is made that the observed sequence was obtained from a continuous time series ($0 < t < (N-1)\Delta$) bandlimited to W hertz and sampled every $\Delta = 1/2W$ seconds.

The motivation for considering several detector structures can be found in the details of any one particular problem. Due to considerations of computational speed or the underlying physics of a given situation, one structure may be preferred over another. For example, discrete Fourier transforms are faster to compute than autocorrelation functions. However, the ability to window the autocorrelation function before transforming to the frequency domain provides a certain degree of flexibility in what the resulting display looks like. Another concern is stability of the sinusoid. If its phase or frequency vary with time, averaging of power spectra computed from short consecutive data sets may be necessary to achieve the best detectability.

The organization of this report is as follows. In Section II, the likelihood ratio processor is defined and its performance evaluation in terms of the ROC (receiver operating characteristic) curve is discussed. The next three sections develop three optimal receiver structures – SKE (signal known exactly), SKEP (signal known except for phase), and SKEP Independent Increments. Section VI defines the discrete Fourier Transform (DFT) and relates it to each of the three optimal detector structures mentioned above. The autocorrelation function and its relationship to the power spectrum of a sequence are the subject of the following section. Lastly, Section VIII develops expressions for the REC (recursive exponential correlator).

II. DETECTION THEORY

A. The Likelihood Ratio

A helpful baseline from which to compute various detector structures is that of the likelihood ratio processor. Let the vector of observables be denoted by

$$\underline{X} = [X(0)\ldots, X(N-1)]^T. \tag{2.1}$$

Based upon the observation vector $\underline{X} \in \chi$, the processor must make a decision (D_0 or D_1) as to which hypothesis it believes is true. Classical detection theory has shown that decisions based upon the likelihood ratio are optimum for a wide range of goodness criteria (Peterson, Birdsall, and Fox, 1954)

$$\Lambda(\underline{X}) \triangleq \frac{p(\underline{X}|H_1)}{p(\underline{X}|H_0)} \underset{D_0}{\overset{D_1}{\gtrless}} \eta. \tag{2.2}$$

Birdsall has shown more generally that any optimality criterion based on "detection probability" $P(D_1|H_1)$ and "false alarm probability" $P(D_1|H_0)$ where good decisions are preferred over bad leads to the calculation of $\Lambda(\underline{X})$ as the decision statistic (Birdsall, 1973). Thus, a separation is achieved between the processing of $\underline{X}$ and the actual optimality criterion chosen which arises in the selection of a threshold value η.

The situation may arise where one or several parameters under either or both H_0 and H_1 are uncertain. These are then modeled as random variables and any prior knowledge about them is summarized by a priori probability density functions $p(\underline{\theta}_0)$ and $p(\underline{\theta}_1)$. The desired decision statistic now becomes the ratio of marginal probability density functions on χ:

$$\Lambda(\underline{X}) = \frac{\int_{\Theta_1} p(\underline{X}|\underline{\theta}_1, H_1)\, p(\underline{\theta}_1)\, d\underline{\theta}_1}{\int_{\Theta_0} p(\underline{X}|\underline{\theta}_0, H_0)\, p(\underline{\theta}_0)\, d\underline{\theta}_0}, \tag{2.3}$$

where $\underline{\theta}_1 \epsilon \Theta_1$ and $\underline{\theta}_0 \epsilon \Theta_0$. A block diagram of the processor structure implied by (2.3) is given in Figure 2.1.

B. Performance

The complete description of a detection device includes both the processor itself (i.e., the mathematical transformation from observation space to decision statistic) and the performance of the processor evaluated with respect to the goodness criterion initially chosen. As mentioned earlier, the likelihood ratio has been shown optimum for any goodness criterion based on "detection probability" $P(D_1|H_1)$ and "false alarm probability $P(D_1|H_0)$ where good decisions are preferred over bad. Thus, the appropriate description of performance for a likelihood ratio computing device is its detection and false alarm probabilities as a function of decision threshold. The precise definition of these terms (which arise from within a RADAR and SONAR context) now will be given.

Since the likelihood ratio is simply a transformation of random variables (the observation vector $\underline{X}$) to a one-dimensional decision statistic ($\Lambda(\underline{X})$), the likelihood ratio itself will be a random variable whose probability density function will depend on which hypothesis (H_0 or H_1) is actually active on χ. Recalling that the threshold η divides the decision space, define

$$P_D \triangleq P(D_1|H_1) \triangleq \int_{\eta}^{\infty} p(\Lambda|H_1)\, d\Lambda \tag{2.4}$$

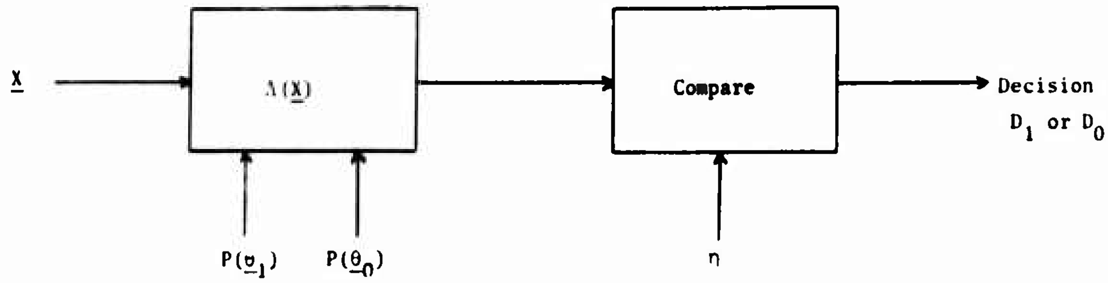


Figure 2.1. Likelihood ratio processor.

$$P_F \triangleq P(D_1|H_0) \triangleq \int_{\eta}^{\infty} p(\Lambda|H_0)\, d\Lambda. \tag{2.5}$$

Peterson, Birdsall, and Fox introduced a graphical representation of P_D versus P_F as a function of η known as the ROC (receiver operating characteristic) curve (Peterson, Birdsall, and Fox, 1954). The ROC curve will be the means by which performance of the detection receivers discussed in this report will be evaluated and compared.

III. SKE (Signal Known Exactly)

In the SKE problem, the received signal (when present) is known exactly. The statistics are Gaussian under both H_1 and H_0.

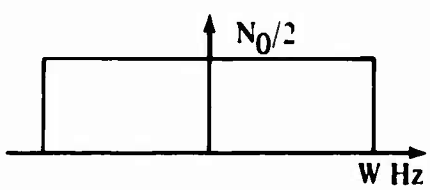
$$\Lambda(\underline{X}) = \frac{p(\underline{X}|H_1)}{p(\underline{X}|H_0)} \qquad \text{(the likelihood ratio)}$$

$$= \frac{\dfrac{1}{(2\pi\sigma^2)^{N/2}} \exp\left[\dfrac{-\sum\limits_{n=0}^{N-1}(X(n)-S(n))^2}{2\sigma^2}\right]}{\dfrac{1}{(2\pi\sigma^2)^{N/2}} \exp\left[\dfrac{-\sum\limits_{n=0}^{N-1}X(n)^2}{2\sigma^2}\right]}$$

$$= \exp\left[\sum_{n=0}^{N-1} \frac{X(n)\,S(N)}{\sigma^2} - \frac{S(n)^2}{2\sigma^2}\right]$$

$$= \exp\left[\frac{1}{\sigma^2}\left(\sum_{n=0}^{N-1} X(n)\,S(n) - \frac{1}{2}\sum_{n=0}^{N-1} S(n)^2\right)\right]. \tag{3.1}$$

Note: Bandlimited white Gaussian Noise



$$\sigma^2 = N_0 W$$

$E \triangleq$ signal energy

$$= \int_0^{N\Delta} S(t)^2\, dt \cong \frac{1}{2W} \sum_{n=0}^{N-1} S(n)^2. \tag{3.2}$$

Thus,

$$\frac{1}{2\sigma^2}\sum_{n=0}^{N-1} S(n)^2 = \frac{1}{2\,N_0W}\sum_{n=0}^{N-1} S(n)^2$$

$$\cong \frac{E}{N_0}. \tag{3.3}$$

$$\Lambda(\underline{X}) = \exp\left[\frac{1}{\sigma^2}\sum_{n=0}^{N-1} X(n)\,S(n) - \frac{E}{N_0}\right]. \tag{3.4}$$

Any monotonic function of $\Lambda(\underline{X})$ will also be an optimal test statistic.

$$\ell n\,\Lambda(\underline{X}) = -\frac{E}{N_0} + \frac{1}{\sigma^2}\sum_{n=0}^{N-1} X(n)\,S(n) \tag{3.5}$$

$$\cong -\frac{E}{N_0} + \frac{2}{N_0}\int_0^{N\Delta} X(t)\,S(t)\,dt. \tag{3.6}$$

The last term in each of the above two expressions is known as a "matched filter."

Substituting the expression for S(n),

$$\sum_{n=0}^{N-1} X(n)\cos(2\pi f_0 n\Delta) = \frac{\sigma^2}{A}\left(\ell n\,\Lambda(\underline{X}) + \frac{E}{N_0}\right). \tag{3.7}$$

Since the parameters A, σ^2, E, and N_0 are known in the SKE problem, the above is simply a monotonic function of $\Lambda(\underline{X})$. Thus, $a = \mathrm{Re}\{X(k)\}$, $k = Nf_0\Delta$ is an optimal decision statistic [see Section VI for the definition of X(k)]. Computing the DFT of the observed sequence will be the optimal processor when the signal is known exactly.

$$E[a|H_1] = \frac{A}{2}N$$

$$E[a|H_0] = 0$$

$$\mathrm{var}[a|H_1] = \mathrm{var}[a|H_0] = \frac{N}{2}\sigma^2.$$

In general, the entire ROC curve is necessary to completely specify performance. However, in the SKE problem, performance is summarized by a single number known as the detectability index d^2. In this case, the distribution of the optimal decision statistic is Gaussian under H_1 and H_0 with equal variances. By definition (Van Trees, 1968)

$$d^2 \triangleq \frac{(E[a|H_1] - E[a|H_0])^2}{\mathrm{var}[a|H_0]}$$

$$= \frac{A^2 N}{2\sigma^2} \qquad \left(= N \cdot SNR;\ SNR = \frac{A^2}{2\sigma^2}\right)$$

$$= \frac{2E}{N_0}. \tag{3.8}$$

Detection and false alarm probability expressions corresponding to Eqs. (2.4) and (2.5) are

$$P_D = \mathrm{erfc}_*\left(\frac{\ln \eta}{d} - \frac{d}{2}\right) \tag{3.9}$$

$$P_F = \mathrm{erfc}_*\left(\frac{\ln \eta}{d} + \frac{d}{2}\right), \tag{3.10}$$

where

$$\mathrm{erfc}_* = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx. \tag{3.11}$$

The SKE performance curves are illustrated in Figure 3.1 on normal-normal paper. Note that performance increases linearly on the negative diagonal as a function of d.

IV. SKEP (Signal Known Except for Phase)

In the SKEP problem, the received signal (when present) is known except for its phase:

$$S(n) = A \cos(2\pi f_0 \Delta + \phi),$$

where: ϕ is a random variable distributed uniformly between $-\pi$ and π.

The statistics are conditionally Gaussian under H_1 and Gaussian under H_0.

$$\Lambda(\underline{X}) = \int_{-\pi}^{\pi} \Lambda(\underline{X}|\phi)\, p(\phi)\, d\phi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left[-\frac{E}{N_0} + \frac{A}{\sigma^2}\sum_{n=0}^{N-1} X(n)\cos(2\pi f_0 n\Delta + \phi)\right] d\phi$$

$$= \exp\left[-\frac{E}{N_0}\right] \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{A}{\sigma^2}(a^2 + b^2)^{1/2} \cos(\theta + \phi)\, d\phi$$

$$= \exp\left[-\frac{E}{N_0}\right] I_0\left[\frac{A}{\sigma^2}(a^2 + b^2)^{1/2}\right]. \tag{4.1}$$

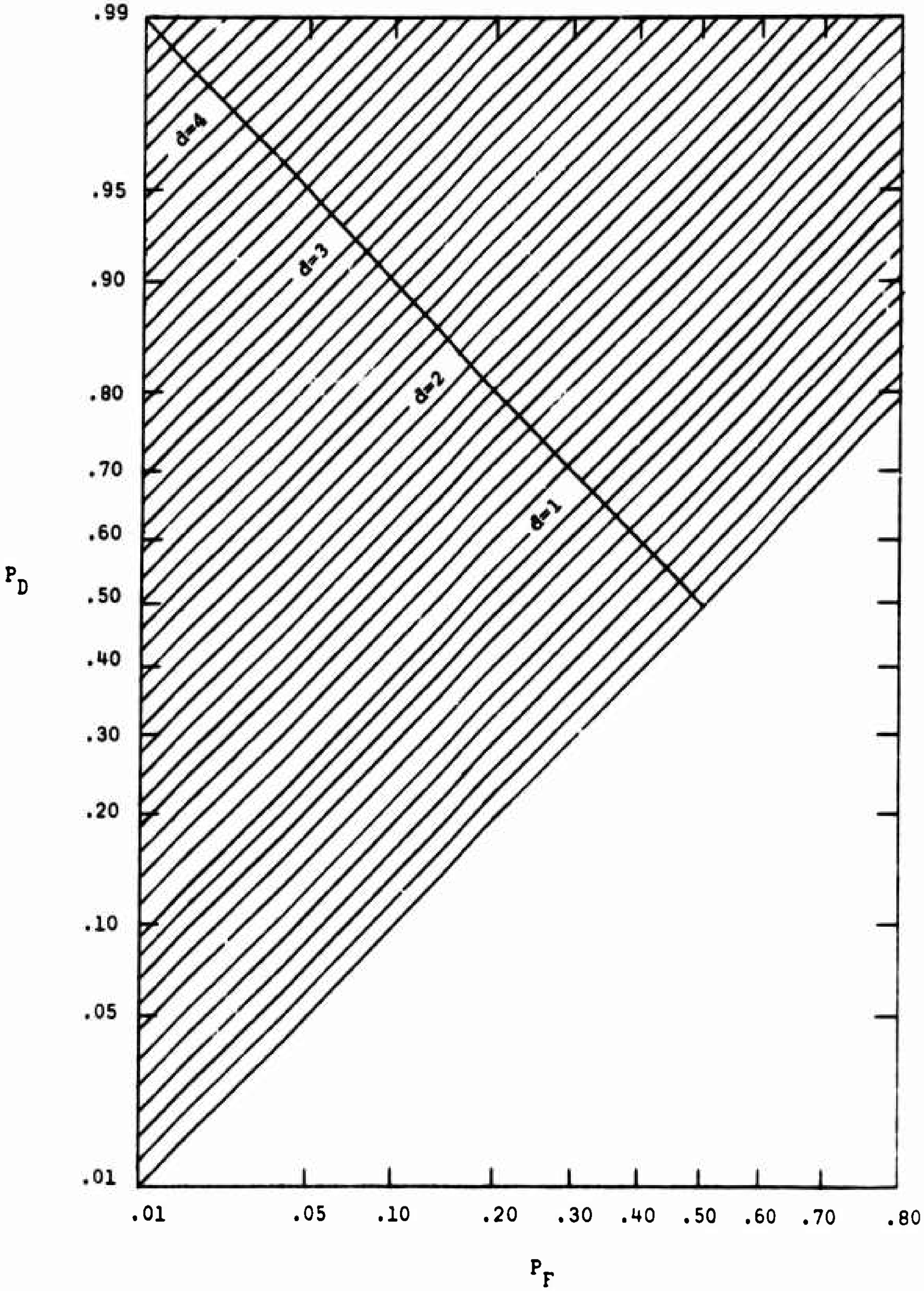


Figure 3.1. Performance of the SKE processor.

where:

$$X(k) = a + ib$$
$$k = Nf_0\Delta$$
$$\theta = \tan^{-1}(b/a).$$

Note the following monotonic function of the likelihood ratio:

$$\Lambda(X) \propto (a^2 + b^2) = |X(k)|^2, \qquad k = Nf_0\Delta. \tag{4.2}$$

Thus, $a^2 + b^2 = |X(k)|^2$ is an optimal decision statistic [see Section VI for definition of $X(k)$]. Computing the power spectrum of the observed sequence is the optimal processor when the signal is known except for phase, with the phase having a uniform distribution.

The ROC curves of the optimal detector for the SKEP problem are given in Figure 4.1 (Roberts, 1965). The loss in performance due to phase uncertainty can be seen by comparing the SKEP curves with the SKE curves for the same value of $2E/N_0$. (See also Van Trees, 1968.)

Two important points must be stressed with respect to this problem. First, it appears that the loss in performance from SKE can be summarized by noting that the test statistic contains twice as much noise (i.e., simply a 3-dB loss). Using this reasoning, the SKEP ROC curve in Figure 4.1 for $2E/N_0 = 9$ roughly should be parallel to the SKE ROC curve for $2E/N_0 = 4$. Note that the two ROC's cross each other, indicating a more fundamental differential in performance than can be accounted for by a simple adjustment of $2E/N_0$.

The second point concerns the use of a detectability index to compare performance between two processors. A detectability index for the SKEP processor can be defined as in Eq. (3.8).

$$I_N(k) = \frac{1}{N}|X(k)|^2, \qquad k = Nf_0\Delta \qquad \text{(See Section VI.)}$$

$$= \frac{1}{N}(a^2 + b^2)$$

$$E[I_N(k)|H_0] = \sigma^2$$

$$E[I_N(k)|H_1] = \sigma^2 + \frac{A^2}{4}N$$

$$\mathrm{var}\,[I_N(k)|H_0] = \sigma^4$$

$$\mathrm{var}\,[I_N(k)|H_1] = \sigma^4 + \sigma^2\frac{A^2}{2}N.$$

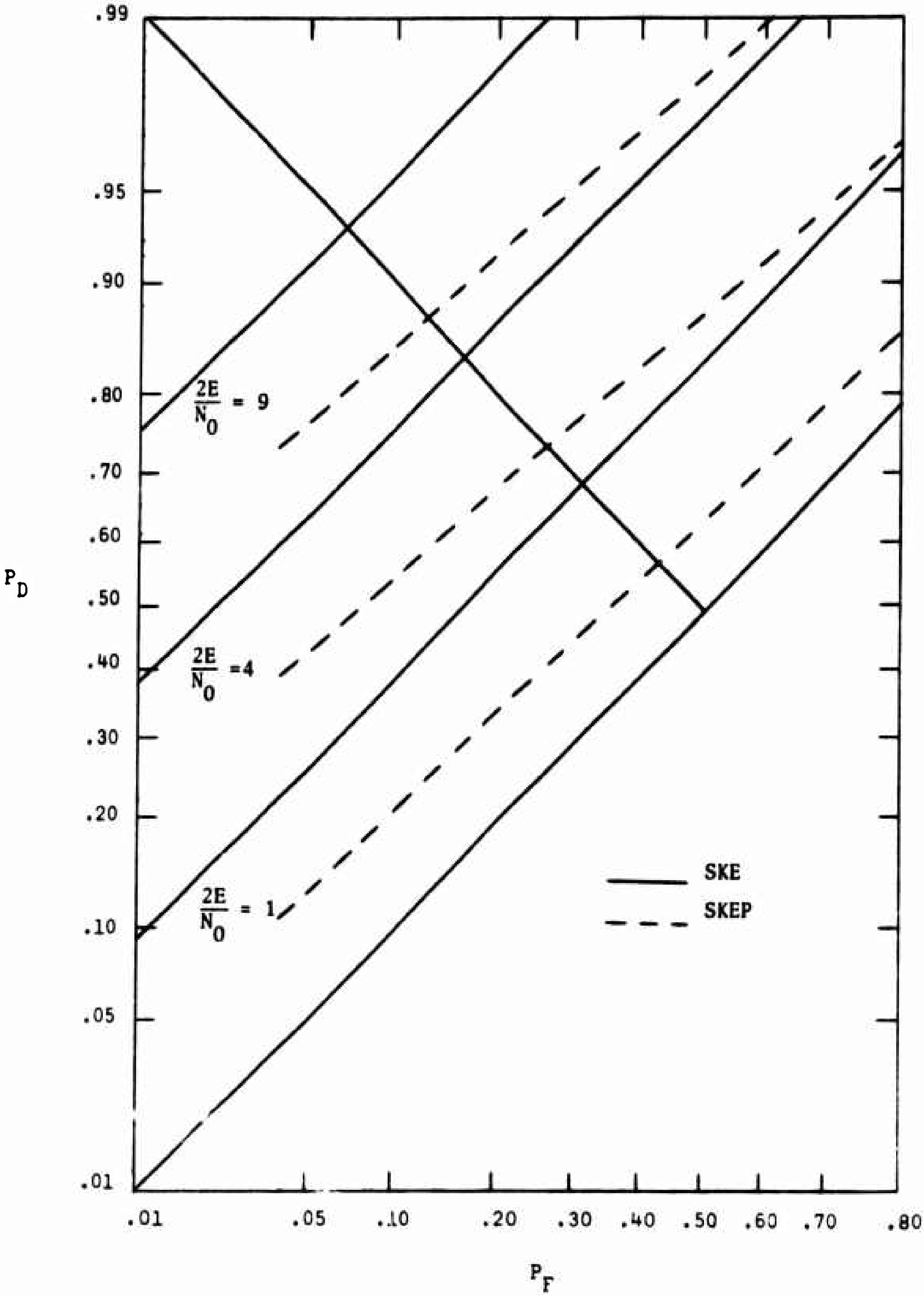


Figure 4.1. Performance of the SKEP processor.

Define:

$$d^2_{SKEP} \triangleq \frac{(E[I_N(k)|H_1] - E[I_N(k)|H_0])^2}{\text{var}\,[I_N(k)|H_0]}$$

$$= \left(\frac{\frac{A^2}{4}N}{\sigma^2}\right)^2 \left(= \left(\frac{N}{2}\cdot \text{SNR}\right)^2\right)$$

$$= \left(\frac{E}{N_0}\right)^2 \tag{4.3}$$

On the basis of Eqs. (3.8) and (4.3) alone, one might incorrectly conclude that the SKEP processor will perform much better than the SKE processor when $E/N_0 > 2$. Recall that the only valid performance comparison is in terms of the ROC curve as shown in Figure 4.1. (In retrospect, it can be seen that Eq. (4.3) does not take into account that var $[I_N(k)|H_1] \neq$ var $[I_N(k)|H_0]$ and that the distributions of the test statistic are now chi-square instead of Gaussian).

V. SKEP (Signal Known Except for Phase) INDEPENDENT INCREMENTS

In the SKEP Independent Increments problem the received signal (when present) is known except for its phase

$$S(n) = A\cos(2\pi f_0 n\Delta + \phi_\ell),$$

where:

ϕ_ℓ is a random variable distributed uniformly between $-\pi$ and π.

ϕ_i is constant for $\ell\cdot N \leq n \leq (\ell+1)N$, $\quad \ell = 0, \ldots, L-1$
but changes independently between increments.

Data sequence consists of L blocks, each of length N. The statistics are conditionally Gaussian under H_1 and Gaussian under H_0.

$$\Lambda(\underline{X}) = \prod_{\ell=0}^{L-1} \Lambda(\underline{X}_\ell)$$

$$= \exp\left[-\frac{L\cdot E}{N_0}\right] \prod_{\ell=0}^{L-1} I_0\left[\frac{A}{\sigma^2}(a_\ell^2 + b_\ell^2)^{1/2}\right], \tag{5.1}$$

where:

$\underline{X}_\ell \;= [X(\ell\cdot N), \ldots, X((\ell+1)\cdot N - 1)]^T$

$X_\ell(k) = a_\ell + ib_\ell, \qquad k = Nf_0\Delta.$

Note the following monotonic function of the likelihood ratio:

$$\sum_{\ell=0}^{L-1} \ell n\, I_0\left[\frac{A}{\sigma^2}\left(a_\ell^2 + b_\ell^2\right)^{1/2}\right] = \ell n\, \Lambda(\underline{X}) + \frac{L \cdot E}{N_0}. \tag{5.2}$$

Furthermore,

$$\ell n\, I_0(x) \cong x - \ell n \sqrt{2\pi x}, \qquad x >> 1$$

$$\cong \frac{x^2}{4}, \qquad x << 1.$$

When

$$N \cdot SNR << 1 \qquad \left[SNR = \left(\frac{A^2}{2\sigma^2}\right)\right]$$

$$\sum_{\ell=0}^{L-1} \ell n\, I_0\left[\frac{A}{\sigma^2}\left(a_\ell^2 + b_\ell^2\right)^{1/2}\right] \cong \frac{A^2}{4\sigma^4} \sum_{\ell=0}^{L-1} \left(a_\ell^2 + b_\ell^2\right). \tag{5.3}$$

Thus, averaging power spectra

$$\frac{1}{L} \sum_{\ell=0}^{L-1} \frac{1}{N} \left(a_\ell^2 + b_\ell^2\right) \tag{5.4}$$

approximately yields an optimal decision statistic for the SKEP independent increments problem when $N \cdot SNR << 1$ (See Section VI).

Performance of the SKEP independent increments processor is given in Figure 5.1 (Thompson, 1972). ROC curves for 1, 5, 10, and 20 independent increments are illustrated. The SKE ROC curve for the same $2E/N_0$ value is included as a reference. Note that L=1 corresponds to the SKEP processor discussed in Section IV. It can be seen easily from Figure 5.1 the dramatic effect signal instability has on detectability. [Performance for the detector structure in Eq. (5.4) for arbitrary SNR can be found in Marcum, 1960.]

VI. DISCRETE FOURIER TRANSFORM

A. Discrete Fourier Transform (DFT)

$$X(k) \triangleq \sum_{n=0}^{N-1} X(n)\, e^{-i(2\pi/N)nk}$$

$$= a + ib; \qquad a,b \sim N\left(0, \frac{N}{2}\sigma^2\right) \text{ under } H_0.$$

$$X(k)X(k)^* = a^2 + b^2$$

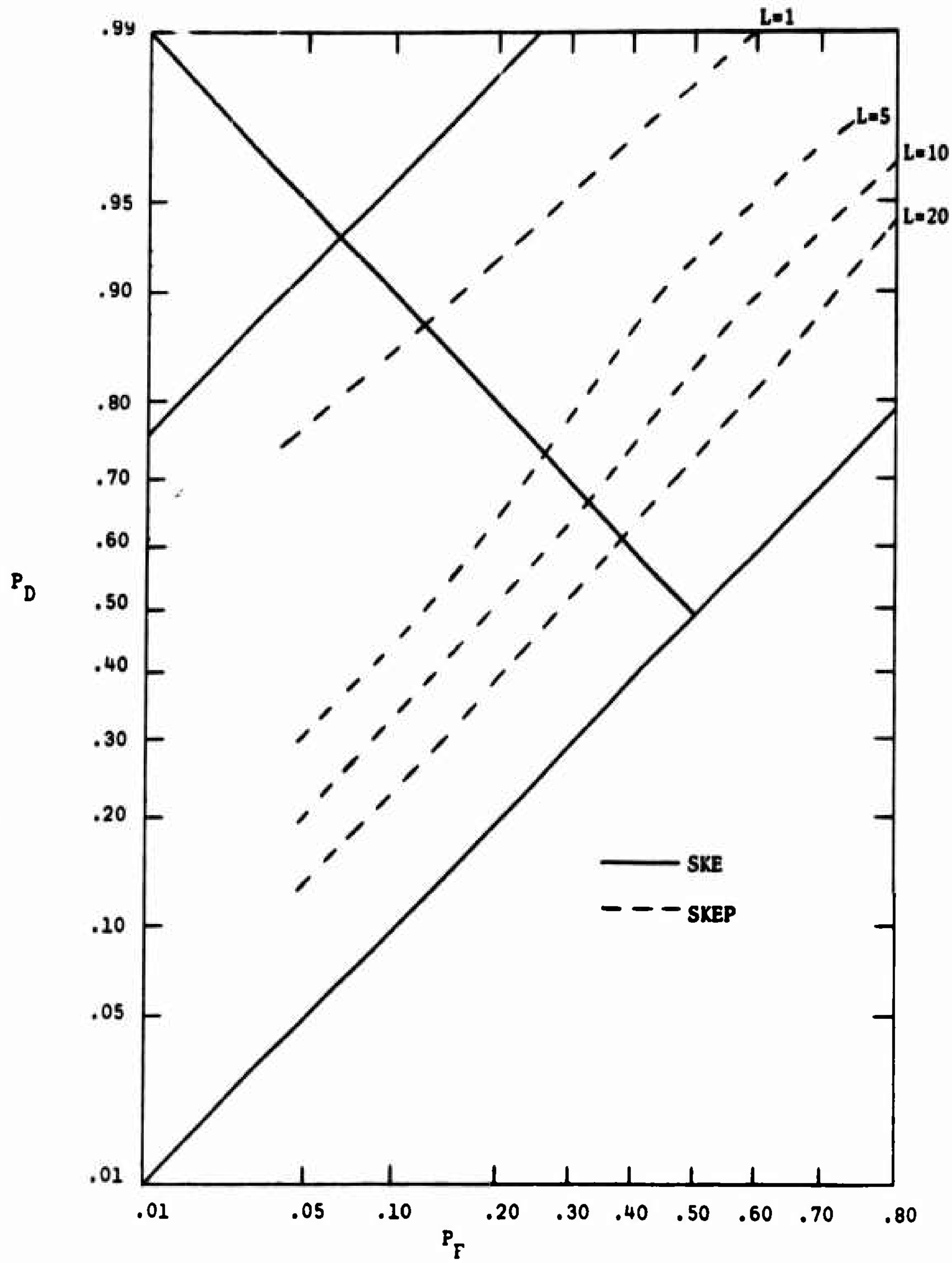


Figure 5.1. Performance of the SKEP Independent Increments Processor. $2E/N_0 = 9.0$.

$$E[X(k)|H_0] = 0$$

$$E[X(k)|H_1] = \frac{A}{2}N; \qquad k = Nf_0\Delta$$

$$= 0; \qquad k \neq Nf_0\Delta$$

$$\text{var}\,[X(k)|H_0] = N\sigma^2$$

$$\text{var}\,[X(k)|H_1] = N\sigma^2$$

Computing the DFT of the observed sequence is the optimal processor when the signal is known exactly (See Section III).

B. Power Spectrum (Periodogram)

$$I_N(k) \triangleq \frac{1}{N}|X(k)|^2$$

$$= \frac{1}{N}(a^2 + b^2); \qquad a, b, \sim N\left(0, \frac{N}{2}\sigma^2\right) \text{ under } H_0.$$

(See also Oppenheim and Schafer, 1975.)

$$E[I_N(k)|H_0] = \sigma^2$$

$$E[I_N(k)|H_1] = \sigma^2 + \frac{A^2}{4}N; \qquad k = Nf_0\Delta$$

$$= \sigma^2; \qquad k \neq Nf_0\Delta$$

$$\text{var}\,[I_N(k)|H_0] = \sigma^4$$

$$\text{var}\,[I_N(k)|H_1] = \sigma^4 + \sigma^2\frac{A^2}{2}N; \qquad k = Nf_0\Delta$$

$$= \sigma^4; \qquad k \neq Nf_0\Delta.$$

Note that the statistics of $I_N(k)$ are chi-square. Computing the power spectrum of the observed sequence is the optimal processor when the signal is known except for phase with the phase having a uniform distribution (See Section IV).

C. Average Power Spectrum

Average K spectra each derived from a time sequence of length N.

$$I_N(k)_{avg} = \frac{1}{K}\sum_{i=0}^{K-1} I_N(k)_i$$

$$= \frac{1}{K}\sum_{i=0}^{K-1} \frac{1}{N}|X(k)_i|^2$$

$$= \frac{1}{K}\sum_{i=0}^{K-1} \frac{1}{N}(a_i^2 + b_i^2)$$

$$E[I_N(k)_{avg}|H_0] = \sigma^2$$

$$E[I_N(k)_{avg}|H_1] = \sigma^2 + \frac{A^2}{4}N; \qquad k = Nf_0\Delta$$

$$= \sigma^2; \qquad k \neq Nf_0\Delta$$

$$\text{var}\,[I_N(k)_{avg}|H_0] = \frac{1}{K}\sigma^4$$

$$\text{var}\,[I_N(k)_{avg}|H_1] = \frac{1}{K}\left\{\sigma^4 + \sigma^2\frac{A^2}{2}N\right\}; \qquad k = Nf_0\Delta$$

$$= \frac{1}{K}\sigma^4; \qquad k \neq Nf_0\Delta.$$

Averaging power spectra computed from consecutive blocks of the observed sequence closely approximates the optimal processor for N·SNR << 1 when the signal is known except for phase, with the phase being uniformly distributed in each increment and independent between increments (See Section V).

VII. AUTOCORRELATION FUNCTION

A. Autocorrelation Function

$$R_{XX}(m) \triangleq \frac{1}{N}\sum_{n=0}^{N-1-m} X(n)\,X(n+m), \qquad m=0, 1, \ldots, N-1.$$

Note: Defining $R_{XX}(-m) = R_{XX}(m)$

$$I_N(k) = \frac{1}{N}|X(k)|^2$$

$$= \sum_{m=-(N-1)}^{N-1} R_{xx}(m)\, e^{-j(2\pi/N)mk}$$

(See Section VI for definition of $I_N(k)$.)

$$E[R_{xx}(m)|H_0] = \sigma^2, \qquad m=0$$

$$= 0, \qquad m>0$$

$$E[R_{xx}(m)|H_1] = \sigma^2 + \frac{A^2}{2}, \qquad m=0$$

$$= \frac{N-m}{N}\frac{A^2}{2}\cos 2\pi f_0 m\Delta, \qquad m>0$$

$$\operatorname{var}[R_{xx}(m)|H_0] = \frac{2}{N}\sigma^4, \qquad m=0$$

$$= \frac{N-m}{N^2}, \qquad m>0$$

$$\operatorname{var}[R_{xx}(m)|H_1] = \,?$$

Note that the statistics of $R_{xx}(m)$ are not Gaussian.

Define:

$$S_{xx}(k,\delta) \triangleq \sum_{m=0}^{L-1} R_{xx}(m+\delta)\, e^{-j(2\pi/L)mk}, \qquad 1 < L < N-\delta.$$

$S_{xx}(k,\delta)$ is the DFT of an L-point segment of the autocorrelation function.

$$E[S_{xx}(k,\delta)|H_0] = \sigma^2, \qquad \delta=0$$

$$= 0, \qquad \delta>0$$

$$E[S_{xx}(k,\delta)|H_1] = \sigma^2 + \frac{A^2}{2}\left(\frac{L}{2} - \frac{L^2}{4N}\right), \qquad \delta=0;\ k=Nf_0\Delta$$

$$= \frac{A^2}{2}\left(\frac{L}{2} - \frac{L^2+2L\delta}{4N}\right), \qquad \delta>0;\ k=Nf_0\Delta$$

$$\text{var}\,[S_{xx}(k,\delta)|H_0] = \sigma^4\left[\frac{1}{N} + \frac{1}{2N^2}(-L^2 + 2NL + L)\right], \qquad \delta = 0$$

$$= \sigma^4\left\{\frac{1}{2N^2}(-L^2 + 2L(N-\delta) + L)\right\}, \qquad \delta > 0$$

$$\text{var}\,[S_{xx}(k,\delta)|H_1] = ?$$

In general, $S_{xx}(k, \delta)$ is complex. When $\delta = 0$, the expression for variance yields twice the value obtained in Section VI B. In practice, only $\text{Re}\{S_{xx}(k, 0)\}$ would be used.

B. Averaged Autocorrelation Function

Average K autocorrelation functions, each derived from a time sequence of length N.

$$R_{xx}(m)_{avg} = \frac{1}{K}\sum_{i=0}^{K-1} R_{xx}(m)_i$$

$$= \frac{1}{K}\sum_{i=0}^{K-1}\frac{1}{N}\sum_{n=0}^{N-1-m} X(n)_i\, X(n+m)_i, \qquad m=0,\ldots,N-1$$

$$E[S_{xx}(k,\delta)|H_0] = \sigma^2, \qquad \delta = 0$$

$$= 0, \qquad \delta > 0$$

$$E[S_{xx}(k,\delta)|H_1] = \sigma^2 + \frac{A^2}{2}\left(\frac{L}{2} - \frac{L^2}{4N}\right), \qquad \delta = 0;\ k = Nf_0\Delta$$

$$= \frac{A^2}{2}\left(\frac{L}{2} - \frac{L^2 + 2L\delta}{4N}\right), \qquad \delta > 0;\ k = Nf_0\Delta$$

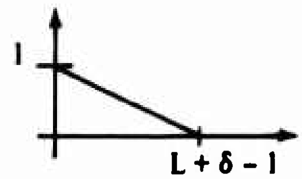
$$\text{var}\,[S_{xx}(k,\delta)|H_0 = \frac{\sigma^4}{K}\left[\frac{1}{N} + \frac{1}{2N^2}\left(-L^2 + 2NL + L\right)\right], \qquad \delta = 0$$

$$= \frac{\sigma^4}{K}\left[\frac{1}{2N^2}\left(-L^2 + 2L(N-\delta) + L\right)\right], \qquad \delta > 0$$

$$\text{var}\,[S_{xx}(k,\delta)|H_1] = ?$$

C. Windowed Autocorrelation Function

Consider the autocorrelation function created from a time sequence of length N. Using a triangular sequence (Bartlett window), window this function back to length $L + \delta$. That is,

$$R_{xx}^{\omega}(m) =$$ 

$$\cdot\, R_{xx}(m)$$

The above is equivalent to defining

$$R_{xx}^{\omega}(m) \triangleq \frac{L+\delta-m}{L+\delta}\,\frac{1}{N}\sum_{n=0}^{N-1-m} X(n)\,X(n+m),\ m = 0, 1, \ldots, L+\delta-1.$$

$$\begin{aligned}
E[R_{xx}^{\omega}(m)|H_0] &= \sigma^2, & m &= 0\\
&= 0, & m &> 0
\end{aligned}$$

$$\begin{aligned}
E[R_{xx}^{\omega}(m)|H_1] &= \sigma^2 + \frac{A^2}{2}, & m &= 0\\
&= \left(\frac{L+\delta-m}{L+\delta}\right)\left(\frac{N-m}{N}\right)\frac{A^2}{2}\cos 2\pi f_0 m\Delta, & m &> 0
\end{aligned}$$

$$\begin{aligned}
\mathrm{var}\,[R_{xx}^{\omega}(m)|H_0] &= \frac{2}{N}\sigma^4, & m &= 0\\
&= \left(\frac{L+\delta-m}{L+\delta}\right)^2\left(\frac{N-m}{N^2}\right)\sigma^4, & m &> 0
\end{aligned}$$

$$\mathrm{var}\,[R_{xx}^{\omega}(m)|H_1] = ?$$

Define:

$$S_{xx}^{\omega}(k,\delta) \triangleq \sum_{m=0}^{L-1} R_{xx}^{\omega}(m+\delta)\, e^{-j(2\pi/L)mk}$$

$$\begin{aligned}
E[S_{xx}^{\omega}(k,\delta)|H_0] &= \sigma^2, & \delta &= 0\\
&= 0, & \delta &> 0
\end{aligned}$$

$$\begin{aligned}
E[S_{xx}^{\omega}(k,\delta)|H_1] &= \sigma^2 + \frac{A^2}{2}\left(\frac{L}{2}-\frac{L}{4}\right), & \delta &= 0;\ k = Nf_0\Delta\\
&= \frac{A^2}{2}\left(\frac{L}{2}-\frac{L^2+2\delta L}{4\,(L+\delta)}\right), & \delta &> 0;\ k = Nf_0\Delta
\end{aligned}$$

$$\mathrm{var}\,[S_{xx}^{\omega}(k,\delta)|H_0] = \sigma^4 \frac{L}{N}\frac{1}{6}\left\{2+\frac{3}{L}+\frac{1}{L^2}\right\}, \qquad \delta = 0;\ N >> L+\delta$$

$$= \sigma^4 \frac{L^3}{(L+\delta)^2 N}\frac{1}{6}\left\{2+\frac{3}{L}+\frac{1}{L^2}\right\}, \qquad \delta > 0;\ N >> L+\delta$$

$$\mathrm{var}\,[S_{xx}^{\omega}(k,\delta)|H_1] = ?$$

Note that windowing (VII C) and averaging (VII B) are not equivalent. Averaging autocorrelation functions implies that the data has been broken into K independent blocks, each of length N. A windowed autocorrelation function is a moving average function across the entire data sequence. Thus, each data sample is allowed to interact with data up to $L + \delta$ lags into the past. In averaging, interaction between data samples is restricted to within the individual data blocks.

VIII. REC (Recursive Exponential Correlator)

Weight vector

$$\underline{W}(j) \triangleq [W_0(j), \ldots, W_{L-1}(j)]^T$$

$$W_m(j+1) = (1-\beta)\,W_m(j) + \beta X(j-\delta-m)\,X(j)$$

$$W_m(j) = \sum_{k=\delta+m}^{j-1} (1-\beta)^{j-1-k}\,\beta X(k-\delta-m)\,X(k); \qquad W_m(0) = 0$$

$$E[W_m(j)|H_0] = \sigma^2\,[1-(1-\beta)^{j-1}], \qquad \delta = m = 0$$

$$= 0, \qquad \delta = 0,\ m > 0$$

$$= 0, \qquad \delta > 0$$

$$E[W_m(j)|H_1] = \frac{A^2}{2}\cos 2\pi f_0(\delta+m)\,[1-(1-\beta)^{j-(\delta+m)-1}], \qquad \delta > 0$$

$$\mathrm{var}\,[W_m(j)|H_0] = \sigma^4\beta^2\left[\frac{1-(1-\beta)^{2(j-(\delta+m)-1)}}{2\beta-\beta^2}\right], \qquad \delta > 0$$

$$\mathrm{var}\,[W_m(j)|H_1] = ?$$

Define:

$$H_j(k) \triangleq \sum_{m=0}^{L-1} W_m(j)\,e^{-j(2\pi/L)mk}$$

$$E[H_j(k)|H_0] = \sigma^2 [1-(1-\beta)^{j-1}], \qquad \delta = 0$$

$$= 0, \qquad \delta > 0$$

$$E[H_j(k)|H_1] = \sigma^2 \{1-(1-\beta)^{j-1}\} + \frac{A^2}{2}\left(\frac{L}{2}-C\right), \qquad \delta = 0$$

$$= \frac{A^2}{2}\left(\frac{L}{2}-C\right) e^{j2\pi f_o \delta \Delta}, \qquad \delta > 0$$

where:

$$C \cong (1-\beta)^{j-1-\delta} \frac{1}{2f_0\Delta}\left[\frac{1-(1-\beta)^{-L}}{1-(1-\beta)^{-1/f_o\Delta}}\right]$$

$$\mathrm{var}\,[H_j(k)|H_0] = \frac{\sigma^4\beta^2}{2\beta-\beta^2}\left\{L-(1-\beta)^{2(j-1-\delta)}\left(\frac{1-(1-\beta)^{-2(L-1)}}{1-(1-\beta)^{-2}}\right)\right\}$$

$$\mathrm{var}\,[H_j(k)|H_1] = ?$$

The REC is similar to the windowed autocorrelation function of Section VII C in that both are moving average filters. In the former, the past is exponentially weighted instead of uniformly as in the latter. Additionally, the REC closely resembles the weight vector of the ALE (adaptive line enhancer) (Widrow, et al., 1975) under the condition of low signal-to-noise ratio.

IX. SUMMARY

The primary emphasis in this report has been the relationship between currently employed methods for the detection of a sinusoid obscured by white noise and three classical problems of detection theory. It was shown that the discrete Fourier transform, the power spectrum, and the averaged power spectrum are each the optimal processor (in the sense of making a least-risk decision) for a particular signal model. Of secondary importance, the first- and second-order statistics under H_1 (signal plus noise) and H_0 (noise alone) were compiled for several related processing schemes of current interest.

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